第3章 3 「複素フーリエ級数」 第3回

解答

1. (1) $-\dfrac{i}{\pi}-\dfrac{2}{\pi^2}$　　(2) $-\dfrac{i}{2\pi}$

(3) $-\dfrac{i}{3\pi}-\dfrac{2}{9\pi^2}$　　(4) $-\dfrac{i}{4\pi}$

2. $$-\frac{1}{4}-\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{i}{n\pi}+\frac{1-(-1)^n}{n^2\pi^2}\right)e^{in\pi x}$$

$$=\cdots+\left(\frac{i}{6\pi}-\frac{1}{9\pi^2}\right)e^{-3\pi xi}+\frac{i}{4\pi}e^{-2\pi xi}$$
$$+\left(\frac{i}{2\pi}-\frac{1}{\pi^2}\right)e^{-\pi xi}-\frac{1}{4}-\left(\frac{i}{2\pi}+\frac{1}{\pi^2}\right)e^{\pi xi}$$
$$-\frac{i}{4\pi}e^{2\pi xi}-\left(\frac{i}{6\pi}+\frac{1}{9\pi^2}\right)e^{3\pi xi}+\cdots$$

解説

1. (1) $\displaystyle\int_{-1}^{0}(-x-1)e^{-i\pi x}\,dx$

$$=\left[-\frac{-x-1}{i\pi}e^{-i\pi x}\right]_{-1}^{0}-\frac{1}{i\pi}\int_{-1}^{0}e^{-i\pi x}\,dx$$
$$=\frac{1}{i\pi}+0-\frac{1}{i\pi}\left[-\frac{1}{i\pi}e^{-i\pi x}\right]_{-1}^{0}$$
$$=\frac{1}{i\pi}-\frac{1}{\pi^2}\left(1-e^{i\pi}\right)$$
$$=-\frac{i}{\pi}-\frac{1}{\pi^2}(1-(-1))$$
$$=-\frac{i}{\pi}-\frac{2}{\pi^2}$$

(2) $\displaystyle\int_{-1}^{0}(-x-1)e^{-i2\pi x}\,dx$

$$=\left[-\frac{-x-1}{i2\pi}e^{-i2\pi x}\right]_{-1}^{0}-\frac{1}{i2\pi}\int_{-1}^{0}e^{-i2\pi x}\,dx$$
$$=\frac{1}{i2\pi}+0-\frac{1}{i2\pi}\left[-\frac{1}{i2\pi}e^{-i2\pi x}\right]_{-1}^{0}$$
$$=\frac{1}{i2\pi}-\frac{1}{4\pi^2}\left(1-e^{i2\pi}\right)$$
$$=-\frac{i}{2\pi}-\frac{1}{4\pi^2}(1-1)$$
$$=-\frac{i}{2\pi}$$

(3) $\displaystyle\int_{-1}^{0}(-x-1)e^{-i3\pi x}\,dx$

$$=\left[-\frac{-x-1}{i3\pi}e^{-i3\pi x}\right]_{-1}^{0}-\frac{1}{i3\pi}\int_{-1}^{0}e^{-i3\pi x}\,dx$$
$$=\frac{1}{i3\pi}+0-\frac{1}{i3\pi}\left[-\frac{1}{i3\pi}e^{-i3\pi x}\right]_{-1}^{0}$$
$$=\frac{1}{i3\pi}-\frac{1}{9\pi^2}\left(1-e^{i3\pi}\right)$$
$$=-\frac{i}{3\pi}-\frac{1}{9\pi^2}(1-(-1))$$
$$=-\frac{i}{3\pi}-\frac{2}{9\pi^2}$$

(4) $\displaystyle\int_{-1}^{0}(-x-1)e^{-i4\pi x}\,dx$

$$=\left[-\frac{-x-1}{i4\pi}e^{-i4\pi x}\right]_{-1}^{0}-\frac{1}{i4\pi}\int_{-1}^{0}e^{-i4\pi x}\,dx$$
$$=\frac{1}{i4\pi}+0-\frac{1}{i4\pi}\left[-\frac{1}{i4\pi}e^{-i4\pi x}\right]_{-1}^{0}$$
$$=\frac{1}{i4\pi}-\frac{1}{16\pi^2}\left(1-e^{i4\pi}\right)$$
$$=-\frac{i}{4\pi}-\frac{1}{16\pi^2}(1-1)$$
$$=-\frac{i}{4\pi}$$

2. $\displaystyle c_0=\frac{1}{2}\int_{-1}^{1}f(x)\,dx=\frac{1}{2}\int_{-1}^{0}(-x-1)\,dx$

$$=\frac{1}{2}\left[-\frac{1}{2}x^2-x\right]_{-1}^{0}=\frac{1}{2}\left(0+\frac{1}{2}-1\right)$$
$$=-\frac{1}{4}$$

$n\neq 0$ のとき

$$c_n=\frac{1}{2}\int_{-1}^{1}f(x)e^{-in\pi x}\,dx$$
$$=\frac{1}{2}\int_{-1}^{0}(-x-1)e^{-in\pi x}\,dx$$
$$=\frac{1}{2}\left\{\left[\frac{x+1}{in\pi}e^{-in\pi x}\right]_{-1}^{0}-\frac{1}{in\pi}\int_{-1}^{0}e^{-in\pi x}\,dx\right\}$$
$$=\frac{1}{2}\left\{\frac{1}{in\pi}-0-\frac{1}{in\pi}\left[-\frac{1}{in\pi}e^{-in\pi x}\right]_{-1}^{0}\right\}$$
$$=\frac{1}{2}\left\{-\frac{i}{n\pi}-\frac{1}{n^2\pi^2}\left(1-e^{in\pi}\right)\right\}$$
$$=\frac{1}{2}\left(-\frac{i}{n\pi}-\frac{1-(-1)^n}{n^2\pi^2}\right)$$

したがって，$f(x)$ の複素フーリエ級数は

$$-\frac{1}{4}-\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{i}{n\pi}+\frac{1-(-1)^n}{n^2\pi^2}\right)e^{in\pi x}$$

$$=\cdots+\left(\frac{i}{6\pi}-\frac{1}{9\pi^2}\right)e^{-3\pi xi}+\frac{i}{4\pi}e^{-2\pi xi}$$
$$+\left(\frac{i}{2\pi}-\frac{1}{\pi^2}\right)e^{-\pi xi}-\frac{1}{4}-\left(\frac{i}{2\pi}+\frac{1}{\pi^2}\right)e^{\pi xi}$$
$$-\frac{i}{4\pi}e^{2\pi xi}-\left(\frac{i}{6\pi}+\frac{1}{9\pi^2}\right)e^{3\pi xi}+\cdots$$