第3章 3 「複素フーリエ級数」 第2回

解答

1. (1) $-2\pi i$ (2) πi

(3) $-\dfrac{2}{3}\pi i$ (4) $\dfrac{1}{2}\pi i$

2. $\displaystyle\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} i e^{inx}$

$$= \cdots + \frac{i}{3}e^{-3xi} - \frac{i}{2}e^{-2xi} + ie^{-xi} - ie^{xi} + \frac{i}{2}e^{2xi} - \frac{i}{3}e^{3xi} + \cdots$$

解説

1. (1) $\displaystyle\int_{-\pi}^{\pi} xe^{-ix}\,dx$

$$= \left[-\frac{x}{i}e^{-ix}\right]_{-\pi}^{\pi} + \frac{1}{i}\int_{-\pi}^{\pi} e^{-ix}\,dx$$

$$= -\frac{1}{i}\left(\pi e^{-i\pi} + \pi e^{i\pi}\right) - \frac{1}{i^2}\left[e^{-ix}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{i}\left(-\pi - \pi\right) + \left(e^{-i\pi} - e^{i\pi}\right)$$

$$= \frac{2\pi}{i} + \left(-1 - (-1)\right)$$

$$= -2\pi i$$

(2) $\displaystyle\int_{-\pi}^{\pi} xe^{-i2x}\,dx$

$$= \left[-\frac{x}{2i}e^{-i2x}\right]_{-\pi}^{\pi} + \frac{1}{2i}\int_{-\pi}^{\pi} e^{-i2x}\,dx$$

$$= -\frac{1}{2i}\left(\pi e^{-i2\pi} + \pi e^{i2\pi}\right) - \frac{1}{4i^2}\left[e^{-i2x}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{2i}\left(\pi + \pi\right) + \frac{1}{4}\left(e^{-i2\pi} - e^{i2\pi}\right)$$

$$= -\frac{2\pi}{2i} + \frac{1}{4}\left(1 - 1\right)$$

$$= \pi i$$

(3) $\displaystyle\int_{-\pi}^{\pi} xe^{-i3x}\,dx$

$$= \left[-\frac{x}{3i}e^{-i3x}\right]_{-\pi}^{\pi} + \frac{1}{3i}\int_{-\pi}^{\pi} e^{-i3x}\,dx$$

$$= -\frac{1}{3i}\left(\pi e^{-i3\pi} + \pi e^{i3\pi}\right) - \frac{1}{9i^2}\left[e^{-i3x}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{3i}\left(-\pi - \pi\right) + \frac{1}{9}\left(e^{-i3\pi} - e^{i3\pi}\right)$$

$$= \frac{2\pi}{3i} + \frac{1}{9}\left(-1 - (-1)\right)$$

$$= -\frac{2}{3}\pi i$$

(4) $\displaystyle\int_{-\pi}^{\pi} xe^{-i4x}\,dx$

$$= \left[-\frac{x}{4i}e^{-i4x}\right]_{-\pi}^{\pi} + \frac{1}{4i}\int_{-\pi}^{\pi} e^{-i4x}\,dx$$

$$= -\frac{1}{4i}\left(\pi e^{-i4\pi} + \pi e^{i4\pi}\right) - \frac{1}{16i^2}\left[e^{-i4x}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{4i}\left(\pi + \pi\right) + \frac{1}{16}\left(e^{-i4\pi} - e^{i4\pi}\right)$$

$$= -\frac{2\pi}{4i} + \frac{1}{16}\left(1 - 1\right)$$

$$= \frac{1}{2}\pi i$$

2. $\displaystyle c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

($\because f(x) = x$ は奇関数)

$n \neq 0$ のとき

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left\{\left[-\frac{x}{ni}e^{-inx}\right]_{-\pi}^{\pi} + \frac{1}{ni}\int_{-\pi}^{\pi} e^{-inx}\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{1}{ni}\left(\pi e^{-in\pi} + \pi e^{in\pi}\right) + \frac{1}{n^2 i^2}\left[e^{-inx}\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{1}{ni}\left((-1)^n\pi + (-1)^n\pi\right) + \frac{1}{n^2}\left(e^{-in\pi} - e^{in\pi}\right)\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{2(-1)^{n+1}\pi}{ni} + \frac{1}{n^2}\left((-1)^n - (-1)^n\right)\right\}$$

$$= \frac{(-1)^n}{n}i$$

したがって，$f(x)$ の複素フーリエ級数は

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} i e^{inx}$$

$$= \cdots + \frac{i}{3}e^{-3xi} - \frac{i}{2}e^{-2xi} + ie^{-xi} - ie^{xi} + \frac{i}{2}e^{2xi} - \frac{i}{3}e^{3xi} + \cdots$$