

第2章 1 「ラプラス変換」 第3回

解答

1. $\frac{s}{s^2+4} \quad (s > 0)$

2. (1) $\frac{2+4s+4s^2}{s^3}$ (2) $\frac{s+4}{(s+2)^3}$ (3) $\frac{s^2+2}{s(s^2+4)}$ (4) $\frac{s+3}{(s+3)^2+9}$ (5) $\frac{6s}{(s^2+9)^2}$

(6) $\frac{2}{s^2-4}$ (7) $\frac{1}{s(s+1)}$ (8) $\log \left| \frac{s-2}{s-3} \right|$ (9) $\frac{s}{s^2-9}$

3. (1) $f(t) = 2U(t-1), \mathcal{L}[f(t)] = \frac{2e^{-s}}{s}$

(2) $f(t) = U(t) + U(t-1) + U(t-2), \mathcal{L}[f(t)] = \frac{1+e^{-s}-e^{-2s}}{s}$

解説

1. $s > 0$ のとき, $\lim_{t \rightarrow \infty} e^{-st} \cos 2t = 0, \lim_{t \rightarrow \infty} e^{-st} \sin 2t = 0$ だから

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \cos 2t \, dt = \left[-\frac{1}{s} e^{-st} \cos 2t \right]_0^{\infty} - \frac{2}{s} \int_0^{\infty} e^{-st} \sin 2t \, dt \\ &= \frac{1}{s} - \frac{2}{s} \left(\left[-\frac{1}{s} e^{-st} \sin 2t \right]_0^{\infty} + \frac{2}{s} \int_0^{\infty} e^{-st} \cos 2t \, dt \right) = \frac{1}{s} - \frac{4}{s^2} F(s) \end{aligned}$$

よって $F(s) + \frac{4}{s^2} F(s) = \frac{1}{s} \quad \therefore \mathcal{L}[\cos 2t] = \frac{s}{s^2+4} \quad (s > 0)$

2. (1) $\mathcal{L}[(t+2)^2] = \mathcal{L}[t^2+4t+4] = \mathcal{L}[t^2] + 4\mathcal{L}[t] + 4\mathcal{L}[1] = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} = \frac{2+4s+4s^2}{s^3}$

(2) $\mathcal{L}[(t^2+t)e^{-2t}] = \mathcal{L}[t^2e^{-2t} + te^{-2t}] = \mathcal{L}[t^2e^{-2t}] + \mathcal{L}[te^{-2t}] = \frac{2}{(s+2)^3} + \frac{1}{(s+2)^2} = \frac{s+4}{(s+2)^3}$

(3) $\mathcal{L}[\cos^2 t] = \mathcal{L}\left[\frac{1+\cos 2t}{2}\right] = \frac{1}{2} (\mathcal{L}[1] + \mathcal{L}[\cos 2t]) = \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2+4} \right) = \frac{s^2+2}{s(s^2+4)}$

(4) $\mathcal{L}[e^{\alpha t} \cos \beta t] = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}$ より $\mathcal{L}[e^{-3t} \cos 3t] = \frac{s+3}{(s+3)^2+9}$

(5) $\mathcal{L}[t \sin \omega t] = \frac{2\omega s}{(s^2+\omega^2)^2}$ より $\mathcal{L}[t \sin 3t] = \frac{6s}{(s^2+3^2)^2} = \frac{6s}{(s^2+9)^2}$

(6) $\mathcal{L}[\sinh \omega t] = \frac{\omega}{s^2-\omega^2}$ より $\mathcal{L}[\sinh 2t] = \frac{2}{s^2-2^2} = \frac{2}{s^2-4}$

(7) $\mathcal{L}\left[\int_0^t f(\tau) \, d\tau\right] = \frac{1}{s} F(s)$ より $\mathcal{L}\left[\int_0^t e^{-\tau} \, d\tau\right] = \frac{1}{s} \frac{1}{s+1} = \frac{1}{s(s+1)}$

(8) $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(\sigma) \, d\sigma$ より $\mathcal{L}\left[\frac{e^{3t}-e^{2t}}{t}\right] = \int_s^{\infty} \left(\frac{1}{\sigma-3} - \frac{1}{\sigma-2} \right) \, d\sigma$
 $= \left[\log|\sigma-3| - \log|\sigma-2| \right]_s^{\infty} = \left[\log \left| \frac{\sigma-3}{\sigma-2} \right| \right]_s^{\infty} = -\log \left| \frac{s-3}{s-2} \right| = \log \left| \frac{s-2}{s-3} \right|$

(9) $\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$ より $\mathcal{L}[\delta(t) * \cosh 3t] = \mathcal{L}[\delta(t)]\mathcal{L}[\cosh 3t] = 1 \cdot \frac{s}{s^2-3^2} = \frac{s}{s^2-9}$

3. $\mathcal{L}[U(t-a)] = \frac{e^{-as}}{s} \quad (a \geq 0)$ である.

(1) $\mathcal{L}[2U(t-1)] = \frac{2e^{-s}}{s}$

(2) $\mathcal{L}[U(t)+U(t-1)+U(t-2)] = \mathcal{L}[U(t)] + \mathcal{L}[U(t-1)] + \mathcal{L}[U(t-2)] = \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1+e^{-s}+e^{-2s}}{s}$