第１章 4 「スカラー場，ベクトル場の線積分，面積分」 第３回

解答

1. $\dfrac{4+\sqrt{2}}{3}$

2. (1) -6　　(2) 10

3. $\dfrac{3\sqrt{3}}{2}$

4. $\dfrac{1}{3}$

解説

1. C_1 上では $\dfrac{d\boldsymbol{r}}{dt} = \left(\dfrac{dx}{dt},\ \dfrac{dy}{dt},\ \dfrac{dz}{dt}\right) = (1,\ 1,\ 0)$

$$\frac{ds}{dt} = \left|\frac{d\boldsymbol{r}}{dt}\right| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$$

C_2 上では $\dfrac{d\boldsymbol{r}}{dt} = \left(\dfrac{dx}{dt},\ \dfrac{dy}{dt},\ \dfrac{dz}{dt}\right) = (0,\ 0,\ 1)$

$$\frac{ds}{dt} = \left|\frac{d\boldsymbol{r}}{dt}\right| = 1$$

$$\int_{C_1+C_2} (xy+z^2)\,ds$$
$$= \int_{C_1} (xy+z^2)\left|\frac{d\boldsymbol{r}}{dt}\right|dt + \int_{C_2} (xy+z^2)\left|\frac{d\boldsymbol{r}}{dt}\right|dt$$
$$= \int_0^1 t^2\cdot\sqrt{2}\,dt + \int_0^1 (1+t^2)\,dt$$
$$= \sqrt{2}\Big[\frac{t^3}{3}\Big]_0^1 + \Big[t+\frac{t^3}{3}\Big]_0^1$$
$$= \frac{\sqrt{2}}{3} + 1 + \frac{1}{3} = \frac{4+\sqrt{2}}{3}$$

2. (1)　曲線 C_1 上で

$\boldsymbol{a} = (4x,\ 6y,\ z) = (4t,\ 0,\ 0)$

また $\dfrac{d\boldsymbol{r}}{dt} = (1,\ 0,\ 0)$

$$\int_{-C_1} \boldsymbol{a}\cdot d\boldsymbol{r} = -\int_{C_1} \boldsymbol{a}\cdot d\boldsymbol{r} = -\int_{-1}^{2} \boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt$$
$$= -\int_{-1}^{2} (4t,\ 0,\ 0)\cdot(1,\ 0,\ 0)\,dt$$
$$= -\int_{-1}^{2} 4t\,dt = -\Big[2t^2\Big]_{-1}^{2} = -6$$

(2) $\displaystyle\int_{C_1+C_2} \boldsymbol{a}\cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{a}\cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{a}\cdot d\boldsymbol{r}$

曲線 C_1 上では (1) と同様であり，曲線 C_2 上では

$\boldsymbol{a} = (4x,\ 6y,\ z) = (8\cos t,\ 12\sin t,\ 0)$,

$\dfrac{d\boldsymbol{r}}{dt} = (-2\sin t, 2\cos t,\ 0)$ だから

$$\int_{C_1} \boldsymbol{a}\cdot d\boldsymbol{r} = 6\ (\because\ (1)\ \text{より})$$

$$\int_{C_2} \boldsymbol{a}\cdot d\boldsymbol{r}$$
$$= \int_0^{\frac{\pi}{2}} (8\cos t,\ 12\sin t,\ 0)\cdot(-2\sin t,\ 2\cos t,\ 0)\,dt$$
$$= \int_0^{\frac{\pi}{2}} 8\sin t\cos t\,dt = 4\int_0^{\frac{\pi}{2}} \sin 2t\,dt$$
$$= 4\Big[-\frac{1}{2}\cos 2t\Big]_0^{\frac{\pi}{2}} = 4$$
$$\therefore\ \int_{C_1+C_2} \boldsymbol{a}\cdot d\boldsymbol{r} = 6+4 = 10$$

3. $\dfrac{\partial\boldsymbol{r}}{\partial u} = (1,\ 0,\ -1),\ \ \dfrac{\partial\boldsymbol{r}}{\partial v} = (0,\ 1,\ -1)$

これより $\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v} = (1,\ 1,\ 1)$

$$\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$
$$\int_S \varphi\,dS = \int_D \varphi\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right|du\,dv$$
$$= \iint_D (-2u-v+3)\cdot\sqrt{3}\,du\,dv$$
$$= \sqrt{3}\int_0^1 \left\{\int_0^1 (-2u-v+3)\,du\right\}dv$$
$$= \sqrt{3}\int_0^1 \Big[-u^2-uv+3u\Big]_0^1 dv$$
$$= \sqrt{3}\int_0^1 (2-v)\,dv$$
$$= \sqrt{3}\Big[2v-\frac{v^2}{2}\Big]_0^1 = \frac{3\sqrt{3}}{2}$$

4. $\dfrac{\partial\boldsymbol{r}}{\partial u} = (1,\ 0,\ -1),\ \ \dfrac{\partial\boldsymbol{r}}{\partial v} = (0,\ 1,\ -1)$ だから

$\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v} = (1,\ 1,\ 1)$

ここで，S の単位法線ベクトル $\boldsymbol{n}$ を z 成分が正の値になる向きにとり　　$\boldsymbol{n} = \dfrac{\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v}}{\left|\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v}\right|}$

S 上で $\boldsymbol{a} = (2u,\ -v,\ 1-u-v)$ だから

$$\int_S \boldsymbol{a}\cdot\boldsymbol{n}\,dS = \iint_D \boldsymbol{a}\cdot\boldsymbol{n}\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right|du\,dv$$
$$= \iint_D \boldsymbol{a}\cdot\left(\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right)du\,dv$$
$$= \iint_D (2u,\ -v,\ 1-u-v)\cdot(1,\ 1,\ 1)\,du\,dv$$
$$= \int_0^1 \left\{\int_0^{1-u} (u-2v+1)\,dv\right\}du$$
$$= \int_0^1 \Big[uv-v^2+v\Big]_0^{1-u} du$$
$$= \int_0^1 \Big\{u(1-u)-(1-u)^2+(1-u)\Big\}du$$
$$= -2\int_0^1 \big(u^2-u\big)\,du$$
$$= -2\Big[\frac{u^3}{3}-\frac{u^2}{2}\Big]_0^1 = \frac{1}{3}$$