第１章 4 「スカラー場，ベクトル場の線積分，面積分」 第２回

解答

1. $\dfrac{74}{5}$，4

2. 84

3. 48

4. $\dfrac{19}{3}$

解説

1. C 上では $x = 2t,\ y = 3t^2,\ z = 3t^3$ だから

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 6t,\ \frac{dz}{dt} = 9t^2$$

$$\begin{aligned}\frac{ds}{dt} &= \left|\frac{d\boldsymbol{r}}{dt}\right| \\ &= \sqrt{2^2 + (6t)^2 + (9t^2)^2} \\ &= \sqrt{(9t^2+2)^2} = 9t^2 + 2\end{aligned}$$

したがって

$$\begin{aligned}\int_C (x+y)\,ds &= \int_C \Big(x(t) + y(t)\Big)\left|\frac{d\boldsymbol{r}}{dt}\right| dt \\ &= \int_{-1}^{1} (2t + 3t^2)(9t^2 + 2)\,dt \\ &= \int_{-1}^{1} (27t^4 + 18t^3 + 6t^2 + 4t)\,dt \\ &= 2\int_0^1 (27t^4 + 6t^2)\,dt\end{aligned}$$

($\because$ 偶関数，奇関数に関する積分の性質)

$$= 2\Big[\frac{27}{5}t^5 + 2t^3\Big]_0^1 = 2 \times \frac{37}{5} = \frac{74}{5}$$

$$\begin{aligned}\int_C (x+y)\,dx &= \int_{-1}^{1} (2t + 3t^2)\,\frac{dx}{dt}\,dt \\ &= \int_{-1}^{1} (2t + 3t^2)\cdot 2\,dt \\ &= \int_{-1}^{1} (4t + 6t^2)\,dt \\ &= 2\int_0^1 6t^2\,dt\end{aligned}$$

($\because$ 偶関数，奇関数に関する積分の性質)

$$= 2\Big[2t^3\Big]_0^1 = 4$$

2. 曲線 C 上で

$$\boldsymbol{a} = (xy,\ yz,\ zx) = (15t^3,\ 12t^2,\ 20t^3)$$

また $\dfrac{d\boldsymbol{r}}{dt} = (10t,\ 3,\ 4)$ だから

$$\begin{aligned}\int_C \boldsymbol{a}\cdot d\boldsymbol{r} &= \int_{-1}^{1} \boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt \\ &= \int_{-1}^{1} (15t^3,\ 12t^2,\ 20t^3)\cdot(10t,\ 3,\ 4)\,dt \\ &= \int_{-1}^{1} (150t^4 + 80t^3 + 36t^2)\,dt \\ &= 2\int_0^1 (150t^4 + 36t^2)\,dt\end{aligned}$$

($\because$ 偶関数，奇関数に関する積分の性質)

$$= 2\Big[30t^5 + 12t^3\Big]_0^1 = 84$$

3. $\dfrac{\partial\boldsymbol{r}}{\partial u} = (0,\ 2,\ 0)$，$\dfrac{\partial\boldsymbol{r}}{\partial v} = (0,\ 0,\ 2)$

これより $\dfrac{\partial\boldsymbol{r}}{\partial u} \times \dfrac{\partial\boldsymbol{r}}{\partial v} = (4,\ 0,\ 0)$

$$\left|\frac{\partial\boldsymbol{r}}{\partial u} \times \frac{\partial\boldsymbol{r}}{\partial v}\right| = 4$$

$$\begin{aligned}\int_S \varphi\,dS &= \int_D \varphi\left|\frac{\partial\boldsymbol{r}}{\partial u} \times \frac{\partial\boldsymbol{r}}{\partial v}\right| du\,dv \\ &= \iint_D 16uv\cdot 4\,du\,dv \\ &= 64\int_0^1 \left\{\int_1^2 uv\,du\right\} dv \\ &= 64\int_0^1 v\left[\frac{u^2}{2}\right]_1^2 dv = 64\cdot\frac{3}{2}\left[\frac{v^2}{2}\right]_0^1 = 48\end{aligned}$$

4. $\dfrac{\partial\boldsymbol{r}}{\partial u} = (1,\ 0,\ -2u)$，$\dfrac{\partial\boldsymbol{r}}{\partial v} = (0,\ 1,\ -2v)$ だから

$$\frac{\partial\boldsymbol{r}}{\partial u} \times \frac{\partial\boldsymbol{r}}{\partial v} = (2u,\ 2v,\ 1)$$

ここで，S の単位法線ベクトル $\boldsymbol{n}$ を z 成分が正の値になる向きにとるから

$$\boldsymbol{n} = \frac{\dfrac{\partial\boldsymbol{r}}{\partial u} \times \dfrac{\partial\boldsymbol{r}}{\partial v}}{\left|\dfrac{\partial\boldsymbol{r}}{\partial u} \times \dfrac{\partial\boldsymbol{r}}{\partial v}\right|}$$

S 上で $\boldsymbol{a} = (2u,\ 2v,\ u+v)$ だから

$$\begin{aligned}\int_S \boldsymbol{a}\cdot\boldsymbol{n}\,dS &= \iint_D \boldsymbol{a}\cdot\boldsymbol{n}\left|\frac{\partial\boldsymbol{r}}{\partial u} \times \frac{\partial\boldsymbol{r}}{\partial v}\right| du\,dv \\ &= \iint_D \boldsymbol{a}\cdot\left(\frac{\partial\boldsymbol{r}}{\partial u} \times \frac{\partial\boldsymbol{r}}{\partial v}\right) du\,dv \\ &= \iint_D (2u,\ 2v,\ u+v)\cdot(2u,\ 2v,\ 1)\,du\,dv \\ &= \int_{-1}^{1}\left\{\int_0^1 (4u^2 + 4v^2 + u + v)\,du\right\} dv \\ &= \int_{-1}^{1}\left[\frac{4}{3}u^3 + 4uv^2 + \frac{1}{2}u^2 + uv\right]_0^1 dv \\ &= \int_{-1}^{1}\left(\frac{4}{3} + 4v^2 + \frac{1}{2} + v\right) dv \\ &= 2\int_0^1 \left(4v^2 + \frac{11}{6}\right) dv\end{aligned}$$

($\because$ 偶関数，奇関数に関する積分の性質)

$$= 2\left[\frac{4}{3}v^3 + \frac{11}{6}v\right]_0^1 = \frac{19}{3}$$