第１章 4 「スカラー場，ベクトル場の線積分，面積分」 第１回

解答

1. 2, $\dfrac{8\sqrt{6}}{15}$

2. $\dfrac{53}{6}$

3. 24π

4. $\dfrac{5}{2}$

解説

1. C 上では $x = t^3,\ y = \dfrac{\sqrt{6}}{2}t^2,\ z = t$ だから

$$\frac{dx}{dt} = 3t^2,\ \frac{dy}{dt} = \sqrt{6}t,\ \frac{dz}{dt} = 1$$

$$\begin{aligned}\frac{ds}{dt} &= \left|\frac{d\boldsymbol{r}}{dt}\right| \\ &= \sqrt{(3t^2)^2 + (\sqrt{6}t)^2 + 1^2} \\ &= \sqrt{9t^4 + 6t^2 + 1} = \sqrt{(3t^2+1)^2} = 3t^2 + 1\end{aligned}$$

したがって

$$\begin{aligned}\int_C (x+z)\,ds &= \int_C \Big(x(t) + z(t)\Big)\left|\frac{d\boldsymbol{r}}{dt}\right| dt \\ &= \int_0^1 (t^3+t)(3t^2+1)\,dt \\ &= \int_0^1 (3t^5 + 4t^3 + t)\,dt \\ &= \left[\frac{t^6}{2} + t^4 + \frac{t^2}{2}\right]_0^1 = 2\end{aligned}$$

$$\begin{aligned}\int_C (x+z)\,dy &= \int_0^1 (t^3+t)\,\frac{dy}{dt}\,dt \\ &= \int_0^1 (t^3+t)\cdot\sqrt{6}t\,dt \\ &= \sqrt{6}\int_0^1 (t^4+t^2)\,dt \\ &= \sqrt{6}\left[\frac{t^5}{5} + \frac{t^3}{3}\right]_0^1 \\ &= \sqrt{6}\times\frac{8}{15} = \frac{8\sqrt{6}}{15}\end{aligned}$$

2. 曲線 C 上で

$\boldsymbol{a} = (yz,\ 2x,\ x+y+z) = (4t^3,\ 2t,\ 5t+t^2)$

また $\dfrac{d\boldsymbol{r}}{dt} = (1,\ 4,\ 2t)$ だから

$$\begin{aligned}\int_C \boldsymbol{a}\cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt \\ &= \int_0^1 (4t^3,\ 2t,\ 5t+t^2)\cdot(1,\ 4,\ 2t)\,dt \\ &= \int_0^1 (6t^3 + 10t^2 + 8t)\,dt \\ &= \left[\frac{3}{2}t^4 + \frac{10}{3}t^3 + 4t^2\right]_0^1 = \frac{53}{6}\end{aligned}$$

3. $\dfrac{\partial\boldsymbol{r}}{\partial u} = (-\sin u,\ \cos u,\ 0)$, $\dfrac{\partial\boldsymbol{r}}{\partial v} = (0,\ 0,\ 2)$

これより $\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v} = (2\cos u,\ 2\sin u,\ 0)$

$$\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right| = 2$$

$$\begin{aligned}\int_S \varphi\,dS &= \int_D \varphi\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right| du\,dv \\ &= \iint_D (\cos^2 u + \sin^2 u + 2v)\cdot 2\,du\,dv \\ &= \int_0^{2\pi}\left\{\int_0^2 (2+4v)\,dv\right\} du \\ &= \int_0^{2\pi}\Big[2v + 2v^2\Big]_0^2 du = \int_0^{2\pi} 12\,du = 24\pi\end{aligned}$$

4. $\dfrac{\partial\boldsymbol{r}}{\partial u} = (1,\ 0,\ -2u)$, $\dfrac{\partial\boldsymbol{r}}{\partial v} = (0,\ 1,\ 0)$ だから

$\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v} = (2u,\ 0,\ 1)$

ここで，S の単位法線ベクトル $\boldsymbol{n}$ を z 成分が正の値になる向きにとるから

$$\boldsymbol{n} = \frac{\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v}}{\left|\dfrac{\partial\boldsymbol{r}}{\partial u}\times\dfrac{\partial\boldsymbol{r}}{\partial v}\right|}$$

S 上で $\boldsymbol{a} = (2u,\ -v,\ v+1-u^2)$ だから

$$\begin{aligned}\int_S \boldsymbol{a}\cdot\boldsymbol{n}\,dS &= \iint_D \boldsymbol{a}\cdot\boldsymbol{n}\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right| du\,dv \\ &= \iint_D \boldsymbol{a}\cdot\left(\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right) du\,dv \\ &= \iint_D (2u,\ -v,\ v+1-u^2)\cdot(2u,\ 0,\ 1)\,du\,dv \\ &= \int_0^1\left\{\int_0^1 (4u^2 + v + 1 - u^2)\,dv\right\} du \\ &= \int_0^1 \left[3u^2 v + \frac{1}{2}v^2 + v\right]_0^1 du \\ &= \int_0^1 \left(3u^2 + \frac{3}{2}\right) du \\ &= \left[u^3 + \frac{3}{2}u\right]_0^1 = \frac{5}{2}\end{aligned}$$