第１章 3 「勾配」「発散」「回転」 第２回

解答

1. $\dfrac{1}{\sqrt{41}}(4,\ 0,\ -5)$

2. (1) 発散 yz，回転 $(0, xy,\ -xz)$

(2) 発散 $e^x + e^{-z}$，回転 $(-1,\ 0,\ 0)$

3. (1) $(2y,\ 2(x+z),\ 2y)$

(2) 0

(3) $(0,\ 2(x+z),\ 0)$

4. $\nabla^2\varphi = 4(x^2+y^2+z^2)$

解説

1. $\nabla\varphi = (4x,\ -6y,\ 10z)$

$(\nabla\varphi)_{\mathrm{P}} = (8,\ 0,\ -10)$

$(\nabla\varphi)_{\mathrm{P}}\cdot\boldsymbol{n}$ が最大となる単位ベクトル $\boldsymbol{n}$ の向きは $(\nabla\varphi)_{\mathrm{P}}$ と同じだから，$(\nabla\varphi)_{\mathrm{P}}$ をその大きさ $|(\nabla\varphi)_{\mathrm{P}}|$ で割ることで単位ベクトルにすればよい．

$|(\nabla\varphi)_{\mathrm{P}}| = 2\sqrt{4^2+0^2+(-5)^2} = 2\sqrt{41}$ より

$$\boldsymbol{n} = \frac{1}{|(\nabla\varphi)_{\mathrm{P}}|}(\nabla\varphi)_{\mathrm{P}} = \frac{1}{2\sqrt{41}}(8,\ 0,\ -10) = \frac{1}{\sqrt{41}}(4,\ 0,\ -5)$$

2. (1) $\nabla\cdot\boldsymbol{a} = \dfrac{\partial}{\partial x}(xyz) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(0) = yz$

$$\nabla\times\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & 0 & 0 \end{vmatrix} = (0,\ xy,\ -xz)$$

(2) $\nabla\cdot\boldsymbol{b} = \dfrac{\partial}{\partial x}(e^x) + \dfrac{\partial}{\partial y}(z) + \dfrac{\partial}{\partial z}(-e^{-z}) = e^x + e^{-z}$

$$\nabla\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & z & -e^{-z} \end{vmatrix} = (-1,\ 0,\ 0)$$

3. (1) $\nabla\cdot\boldsymbol{a} = \dfrac{\partial}{\partial x}(x^2y) + \dfrac{\partial}{\partial y}(-2xz) + \dfrac{\partial}{\partial z}(yz^2) = 2xy + 2yz$

$$\therefore\ \nabla(\nabla\cdot\boldsymbol{a}) = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)(2xy+2yz) = (2y,\ 2(x+z),\ 2y)$$

(2) $\nabla\cdot(\nabla\times\boldsymbol{a}) = 0$

もしくは 詳細に計算すると次のようになる．

$$\nabla\times\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & -2xz & yz^2 \end{vmatrix} = (z^2+2x,\ 0,\ -2z-x^2)$$

$\therefore\ \nabla\cdot(\nabla\times\boldsymbol{a})$

$$= \frac{\partial}{\partial x}(z^2+2x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-2z-x^2) = 2+0-2 = 0$$

(3) (2) より

$\nabla\times\boldsymbol{a} = (2x+z^2,\ 0,\ -x^2-2z)$ だから

$\nabla\times(\nabla\times\boldsymbol{a})$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+z^2 & 0 & -x^2-2z \end{vmatrix} = (0,\ 2(x+z),\ 0)$$

4. $\varphi_x = 2xy^2 + 2z^2x$

$\varphi_y = 2x^2y + 2yz^2$

$\varphi_z = 2y^2z + 2zx^2$

$\varphi_{xx} = 2y^2 + 2z^2$

$\varphi_{yy} = 2x^2 + 2z^2$

$\varphi_{zz} = 2y^2 + 2x^2$

したがって，φ のラプラシアンは

$$\nabla^2\varphi (= \nabla\cdot\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = (2y^2+2z^2) + (2x^2+2z^2) + (2y^2+2x^2) = 4(x^2+y^2+z^2)$$