第１章 3 「勾配」「発散」「回転」 第１回

解答

1. (1) $\nabla\varphi=(4x,\ -6y,\ 10z)$

$(\nabla\varphi)_{\mathrm{P}}=(8,\ 0,\ -10)$

(2) $\dfrac{-2\sqrt{3}}{3}$

2. (1) 発散 $2xy^2-4xz^2$,

回転

$(x^3+3xy^2+8xyz,\ -3x^2y-y^3,\ -4yz^2-2x^2y)$

(2) 発散 e^x，回転 $(e^{-y}+2z,\ 0,\ 0)$

3. (1) $(2y,\ 2x+6yz^2,\ 6y^2z)$

(2) 0

(3) $(0,\ 6yz^2-4x^2-4z^2+2x,\ -2z^3)$

4. $\nabla^2\varphi=2(x+y+z+x^2y^2+y^2z^2+z^2x^2)$

解説

1. (1) $\nabla\varphi=\left(\dfrac{\partial\varphi}{\partial x},\ \dfrac{\partial\varphi}{\partial y},\ \dfrac{\partial\varphi}{\partial z}\right)$

$=(4x,\ -6y,\ 10z)$

$\nabla\varphi$ に点 P の座標値を代入して

$(\nabla\varphi)_{\mathrm{P}}=(8,\ 0,\ -10)$

(2) $(\nabla\varphi)_{\mathrm{P}}\cdot\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=(8,\ 0,\ -10)\cdot\dfrac{1}{\sqrt{3}}(1,\ 1,\ 1)$

$=\dfrac{8}{\sqrt{3}}+0-\dfrac{10}{\sqrt{3}}=-\dfrac{2}{\sqrt{3}}=-\dfrac{2\sqrt{3}}{3}$

2. (1) $\nabla\cdot\boldsymbol{a}=\dfrac{\partial}{\partial x}(x^2y^2)+\dfrac{\partial}{\partial y}(-4xyz^2)$

$+\dfrac{\partial}{\partial z}(x^3y+xy^3)=2xy^2-4xz^2$

$$\nabla\times\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\ x^2y^2&-4xyz^2&x^3y+xy^3\end{vmatrix}$$

$=(x^3+3xy^2+8xyz,$

$-3x^2y-y^3,\ -4yz^2-2x^2y)$

(2) $\nabla\cdot\boldsymbol{b}=\dfrac{\partial}{\partial x}(e^x)+\dfrac{\partial}{\partial y}(-z^2)+\dfrac{\partial}{\partial z}(-e^{-y})$

$=e^x$

$$\nabla\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\ e^x&-z^2&-e^{-y}\end{vmatrix}$$

$=(e^{-y}+2z,\ 0,\ 0)$

3. (1) $\nabla\cdot\boldsymbol{a}=\dfrac{\partial}{\partial x}(x^2y)+\dfrac{\partial}{\partial y}(2x^2z^2)+\dfrac{\partial}{\partial z}(y^2z^3)$

$=2xy+3y^2z^2$

$\therefore\ \nabla(\nabla\cdot\boldsymbol{a})=\left(\dfrac{\partial}{\partial x},\ \dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial z}\right)(2xy+3y^2z^2)$

$=(2y,\ 2x+6yz^2,\ 6y^2z)$

(2) $\nabla\cdot(\nabla\times\boldsymbol{a})=0$

もしくは 詳細に計算すると次のようになる．

$$\nabla\times\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\ x^2y&2x^2z^2&y^2z^3\end{vmatrix}$$

$=(2yz^3-4x^2z,\ 0,\ 4xz^2-x^2)$

$\therefore\nabla\cdot(\nabla\times\boldsymbol{a})$

$=\dfrac{\partial}{\partial x}(2yz^3-4x^2z)+\dfrac{\partial}{\partial y}(0)+\dfrac{\partial}{\partial z}(4xz^2-x^2)$

$=-8xz+0+8xz=0$

(3) (2) より

$\nabla\times\boldsymbol{a}=(2yz^3-4x^2z,\ 0,\ 4xz^2-x^2)$ だから

$\nabla\times(\nabla\times\boldsymbol{a})$

$$=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\ 2yz^3-4x^2z&0&4xz^2-x^2\end{vmatrix}$$

$=(0,\ 6yz^2-4x^2-4z^2+2x,\ -2z^3)$

4. $\varphi_x=2xy^2z^2+2xy+z^2$

$\varphi_{xx}=2y^2z^2+2y$

$\varphi_y=2x^2yz^2+x^2+2yz$

$\varphi_{yy}=2x^2z^2+2z$

$\varphi_z=2x^2y^2z+y^2+2zx$

$\varphi_{zz}=2x^2y^2+2x$

したがって，φ のラプラシアンは

$\nabla^2\varphi(=\nabla\cdot\nabla\varphi)=\dfrac{\partial^2\varphi}{\partial x^2}+\dfrac{\partial^2\varphi}{\partial y^2}+\dfrac{\partial^2\varphi}{\partial z^2}$

$=(2y^2z^2+2y)+(2x^2z^2+2z)+(2x^2y^2+2x)$

$=2(x+y+z+x^2y^2+y^2z^2+z^2x^2)$