第1章 2 「曲線」「曲面」 第3回

解答

1. $\boldsymbol{t} = \left(\dfrac{1-2\cos 2t}{3},\ \dfrac{2}{3}\sin 2t,\ \dfrac{2\sqrt{2}}{3}\cos t \right), \quad s = \dfrac{3}{2}\pi$

2. $\boldsymbol{t} = \left(\dfrac{\sqrt{2}t}{1+t^2},\ \dfrac{\sqrt{2}}{2},\ \dfrac{\sqrt{2}(1-t^2)}{2(1+t^2)} \right), \quad s = 4\sqrt{2}$

3. (1) $\left(\dfrac{u}{\sqrt{1-u^2}},\ 0,\ 1 \right)$ (2) $\dfrac{1}{\sqrt{1-u^2}}$ (3) $\pm(u,\ 0,\ \sqrt{1-u^2})$ (複号同順) (4) $\dfrac{\pi}{2}$

解説

1. $\boldsymbol{r}' = (\,1-2\cos 2t,\ 2\sin 2t,\ 2\sqrt{2}\cos t\,)$

$$
\begin{aligned}
|\boldsymbol{r}'| &= \sqrt{(1-2\cos 2t)^2 + (2\sin 2t)^2 + (2\sqrt{2}\cos t)^2} \\
&= \sqrt{1 - 4\cos 2t + 4\cos^2 2t + 4\sin^4 2t + 8\cos^2 t} \\
&= \sqrt{1 - 4\cos 2t + 4 + 8\cos^2 t} \\
&= \sqrt{5 - 4(2\cos^2 t - 1) + 8\cos^2 t} \\
&\quad (\because\ \cos 2t = 2\cos^2 t - 1\,) \\
&= \sqrt{5 - 8\cos^2 t + 4 + 8\cos^2 t} \\
&= \sqrt{9} = 3
\end{aligned}
$$

$$
\boldsymbol{t} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{1}{3}(\,1-2\cos 2t,\ 2\sin 2t,\ 2\sqrt{2}\cos t\,) = \left(\frac{1-2\cos 2t}{3},\ \frac{2}{3}\sin 2t,\ \frac{2\sqrt{2}}{3}\cos t \right)
$$

$$
s = \int_0^{\frac{\pi}{2}} \left| \frac{d\boldsymbol{r}}{dt} \right| dt = 3\int_0^{\frac{\pi}{2}} dt = \frac{3}{2}\pi
$$

2. $\boldsymbol{r}' = \left(6t,\ 3t^2+3,\ -3t^2+3\right)$

$$
\begin{aligned}
|\boldsymbol{r}'| &= \sqrt{(6t)^2 + (3t^2+3)^2 + (-3t^2+3)^2} \\
&= \sqrt{36t^2 + 9t^4 + 18t^2 + 9 + 9t^4 - 18t^2 + 9} \\
&= \sqrt{18\,(t^4 + 2t^2 + 1)} \\
&= \sqrt{18(t^2+1)^2} = 3\sqrt{2}(t^2+1)
\end{aligned}
$$

$$
\boldsymbol{t} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{1}{3\sqrt{2}(1+t^2)}\left(6t,\ 3t^2+3,\ -3t^2+3\right) = \left(\frac{\sqrt{2}t}{1+t^2},\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}(1-t^2)}{2(1+t^2)} \right)
$$

$$
s = \int_0^1 \left| \frac{d\boldsymbol{r}}{dt} \right| dt = \int_0^1 3\sqrt{2}\left(t^2+1\right)\,dt = 3\sqrt{2}\Big[\,\frac{t^3}{3} + t\,\Big]_0^1 = 3\sqrt{2}\cdot\frac{4}{3} = 4\sqrt{2}
$$

3. (1) $\dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v} = \left(1,\ 0,\ \dfrac{-u}{\sqrt{1-u^2}}\right) \times (0,\ 1,\ 0) = \left(\dfrac{u}{\sqrt{1-u^2}},\ 0,\ 1 \right)$

(2) $\left| \dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v} \right| = \sqrt{\left(\dfrac{u}{\sqrt{1-u^2}} \right)^2 + 0^2 + 1^2} = \sqrt{\dfrac{u^2}{1-u^2} + 1} = \dfrac{1}{\sqrt{1-u^2}}$

(3) $\boldsymbol{n} = \pm\dfrac{\dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v}}{\left| \dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v} \right|} = \pm(u,\ 0,\ \sqrt{1-u^2})$ (複号同順)

(4) $S = \displaystyle\iint_D \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| du\,dv$

$$
= \int_0^1 \left\{ \int_0^1 \frac{1}{\sqrt{1-u^2}}\,dv \right\} du = \int_0^1 \frac{1}{\sqrt{1-u^2}} \Big[\,v\,\Big]_0^1 du = \int_0^1 \frac{1}{\sqrt{1-u^2}}\,du = \Big[\,\sin^{-1} u\,\Big]_0^1 = \frac{\pi}{2}
$$