

解答

1. $(7, -14, -7), \pm \frac{1}{\sqrt{6}}(1, -2, -1)$

2. 垂直な単位ベクトル $\pm \frac{1}{\sqrt{2}}(1, 0, -1)$, $\triangle ABC$ の面積 $4\sqrt{2}$

3. (1) $(1, 1, 2)$ (2) 3

4. (1) $\left(2t, \frac{1}{\sqrt{t}}, 2e^{2t}\right), (2, 1, 2e^2)$ (2) $\left(-2\sqrt{2}\sin t, 2\cos t, \frac{1}{\cos^2 t}\right), (-2, \sqrt{2}, 2)$

5. (1) $4t^3$ (2) $6t+6$ (3) $(-9t^2, 4t^3, 3-4t)$

解説

1. $\mathbf{a} \times \mathbf{b} = (-1 \cdot 5 - 4 \cdot (-3), 4 \cdot (-1) - 2 \cdot 5, 2 \cdot (-3) - (-1) \cdot (-1)) = (7, -14, -7) = 7(1, -2, -1)$

$|\mathbf{a} \times \mathbf{b}| = 7\sqrt{1^2 + (-2)^2 + (-1)^2} = 7\sqrt{6}$ より, \mathbf{a}, \mathbf{b} の両方に垂直な単位ベクトルは

$$\pm \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \pm \frac{7}{7\sqrt{6}}(1, -2, -1) = \pm \frac{1}{\sqrt{6}}(1, -2, -1)$$

2. $\overrightarrow{AB} = (-1, 2, -1), \overrightarrow{AC} = (2, 4, 2)$ より

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-1, 2, -1) \times (2, 4, 2) = (2 \cdot 2 - (-1) \cdot 4, (-1) \cdot 2 - (-1) \cdot 2, (-1) \cdot 4 - 2 \cdot 2) = (8, 0, -8)$$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = 8\sqrt{1^2 + 0^2 + (-1)^2} = 8\sqrt{2}$ より, \overrightarrow{AB} と \overrightarrow{AC} の両方に垂直な単位ベクトルは

$$\pm \frac{8}{8\sqrt{2}}(1, 0, -1) = \pm \frac{1}{\sqrt{2}}(1, 0, -1)$$

また, $\triangle ABC$ の面積は $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$ より, $\frac{1}{2} \cdot 8\sqrt{2} = 4\sqrt{2}$

3. (1) $\mathbf{a} \times \mathbf{b} = (1 \cdot 2 - 4 \cdot 0, 4 \cdot 1 - 2 \cdot 2, 2 \cdot 0 - 1 \cdot 1) = (2, 0, -1)$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (0 \cdot (-1) - (-1) \cdot 1, (-1) \cdot 1 - 2 \cdot (-1), 2 \cdot 1 - 0 \cdot 1) = (1, 1, 2)$$

(2) $\mathbf{b} \times \mathbf{c} = (0 \cdot (-1) - 2 \cdot 1, 2 \cdot 1 - 1 \cdot (-1), 1 \cdot 1 - 0 \cdot 1) = (-2, 3, 1)$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (2, 1, 4) \cdot (-2, 3, 1) = 2 \cdot (-2) + 1 \cdot 3 + 4 \cdot 1 = -4 + 3 + 4 = 3$$

4. (1) $\mathbf{a}'(t) = \left(2t, \frac{1}{\sqrt{t}}, 2e^{2t}\right)$, $t=1$ を代入して, $\mathbf{a}'(1) = (2, 1, 2e^2)$

(2) $\mathbf{b}'(t) = \left(-2\sqrt{2}\sin t, 2\cos t, \frac{1}{\cos^2 t}\right)$, $t = \frac{\pi}{4}$ を代入して, $\mathbf{b}'\left(\frac{\pi}{4}\right) = (-2, \sqrt{2}, 2)$

5. (1) $\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = 2\mathbf{a}' \cdot \mathbf{a} = 2(0, 0, 2t) \cdot (1, 2, t^2) = 4t^3$

(2) $\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}' \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}'$

$$= (0, 0, 2t) \cdot (t^2, 3t, 2) + (1, 2, t^2) \cdot (2t, 3, 0) = 4t + 2t + 6 = 6t + 6$$

(3) $\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}' = (0, 0, 2t) \times (t^2, 3t, 2) + (1, 2, t^2) \times (2t, 3, 0)$

$$= (-6t^2, 2t^3, 0) + (-3t^2, 2t^3, 3-4t) = (-9t^2, 4t^3, 3-4t)$$